

Modeling Dense Stellar Systems

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Black holes and neutron stars present extreme forms of matter that cannot be created as such in a laboratory on Earth. Instead, we have to observe and analyze the experiments that are ongoing in the Universe. The most telling observations of black holes and neutron stars come from dense stellar systems, where stars are crowded close enough to each other to undergo frequent interactions. It is the interplay between black holes, neutron stars and other objects in a dense environment that allows us to use observations to draw firm conclusions about the properties of these extreme forms of matter, through comparisons with simulations. The art of modeling dense stellar systems through computer simulations forms the main topic of this review.

§1. Invitation

If you are a student looking for a thesis topic, or a researcher looking for a new field to explore, you may be interested to consider a relatively new area of study, dense stellar systems. Defined as regions where stars are so close together that they frequently collide, dense stellar systems give rise to all kinds of interesting phenomena. Here are some reasons to consider dense stellar systems:

- if you are interested in fundamental physics, and in extreme forms of matter such as neutron stars and black holes; and if you would like to know where to find them, how to observe them, and how to interpret those observations.
- if you are interested in making fundamental contributions to astrophysics, and you are afraid that all basic discoveries have been made already, then consider dense stellar systems, as an interdisciplinary field where many of the basic questions have not even been addressed, let alone answered.
- if you share Newton's interest in the classical N-body problem, something he didn't have the computational tools for, dense stellar systems offer you the closest application in the real world for this abstract problem, fascinating in its simplicity of formulation, as well as its complexity of behavior.

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As an example of the second point, we can take stellar evolution. The most basic stellar evolution calculations using electronic computers were first performed in the 1950s, and then developed in great detail in the 1960s. The next four decades have mainly seen refinements of modeling techniques, including the difficult treatment of mass transfer in binary systems, but the text books of the 1960s still form a good entry point for learning about the basic approach to the evolution of isolated single stars and normal binary systems.

In contrast, a full stellar evolution modeling of a dense stellar system has yet to be carried out. The very first attempt in that direction was made recently in a PhD thesis by Ross Church in 2006.¹⁾ With respect to evolving the ecological system of an interacting star cluster, we are in a similar state as where Martin Schwarzschild was in the early 1950s, when he first started to use John von Neumann's computer in Princeton to follow the evolution of a single star.²⁾ In short, the exploration of the new frontier of dense stellar systems has just begun and is inviting you to join in the adventure.

The structure of this review is as follows. Section 2 offers five different perspectives on dense stellar systems, from the point of view of fundamental physics, astrophysics, classical physics, computational physics, and interdisciplinary physics collaborations. Section 3 describes how neutron stars and black holes can be detected through radiation emitted from their vicinity. Section 4 introduces different places in the Universe where dense stellar systems can be found, such as globular star clusters and galactic nuclei, with an emphasis on our own galactic center. Section 5 highlights the multi-scale and multi-physics challenges that simulations of dense stellar systems face, and also mentions the use of GRAPE special purpose hardware.

§2. Perspectives on Dense Stellar Systems

Right at the center of our galaxy, a massive black hole resides, surrounded by a dense cluster of stars. To get a sense of how crowded the central region is, let us consider the contents of the inner parsec (pc) of our galaxy, an area that is also called the central nucleus. For comparison, in the neighborhood of our Sun, the distance between individual stars is typically more than 1 pc ($\approx 3 \times 10^{16}$ m). In contrast, a sphere with a radius of 1 pc around the center of our galaxy contains a black hole with a mass of more than three million solar masses, together with a similar amount of mass in stars that move in tight orbits around the central black hole.

The density of stars in the center of our galaxy is thus more than a million times higher than that in the neighborhood of our Sun. And there are other places as well, that have a much higher density than our local neighborhood. The centers of some globular clusters, too, approach a similar density. In such cases, it is unavoidable that many stars undergo close encounters and even physical collisions, with high probability, within their life times. It is these environments, called dense stellar systems, that we will focus on in this review. To start with, we will look at these systems from a variety of different perspectives.

Our Sun has not always been as isolated as it is now. Most likely, it was born in a much denser ‘nest’ of stars. Recently, direct evidence for the formation of the

Sun in a dense stellar system has been obtained by isotope analysis of meteorites in the solar system, hinting at the presence of at least one supernova very close to the young Sun.³⁾

2.1. *Fundamental Physics Perspective*

Black holes constitute the most extreme form of matter known in the universe. According to the best tested theory of gravity that we have, general relativity, the matter in a black hole is compressed in a central singularity, at an infinitely high density. Most likely, this description is only an approximation: whenever a theory in physics predicts the occurrence of singularities, it is a sign that the theory itself breaks down, and has to be replaced by a more detailed description, more appropriate for the area under concern.

Although there are many speculations as to the type of quantum gravity theory that might replace general relativity, we do not yet know which theory is the correct one. Therefore, astrophysicists continue to rely on general relativity as their best guide. There is an additional reason to do so: the disagreements between a quantum gravity theory and general relativity are likely to be confined to an area very close to where general relativity predicts a central singularity. This area lies deep within the event horizon, the area from which no light and no other classical form of information can escape.

As a consequence, effects that are observationally accessible have to occur outside the event horizon, where the classical approximation is expected to be highly accurate. Even so, observational tests are very important in this regime. So far, general relativity has been tested largely in the weak-field approximation. Observations of phenomena just outside the horizon are still mostly unexplored, and would form a very welcome addition to test our most basic theory of gravitation.

Neutron stars are another example of extreme objects. Unlike black holes, they are made out of conventional matter, but in a very extreme form. The density of a neutron star is comparable to the density of an atomic nucleus. Since the diameter of a nucleus is about 10^{-5} of the diameter of an atom, the density of such material is roughly 10^{15} times larger than that of water. If the mass of the Sun would be compressed to form a neutron star, its diameter would be about 10 km, only a few times larger than the Schwarzschild radius of the Sun which is roughly 3 km.

Both black holes and neutron stars can generate copious gravitational waves when they collide and merge. In order to predict the frequency and characteristics of such merger events, simulations of dense clusters of stars systems play an essential role in the ongoing efforts at detections of gravitational waves. In general, all these phenomena, from nuclear matter in bulk to event horizons and gravitational waves, cannot be created in a laboratory on Earth. Therefore, we have to make do with the laboratories that nature provides us, in the form of dense stellar systems, which leads us to a switch from fundamental physics to astrophysics.

2.2. *Astrophysics Perspective*

When massive stars undergo a supernova explosion at the end of their life, they may produce a black hole or a neutron star as a remnant. Such a remnant is difficult

to observe, a black hole because it is black, a neutron star because its size is so small that the thermal radiation emitted well after its birth is hard to detect. Depending on its magnetic field and spin rate, a neutron star can be visible at radio wavelengths as a pulsar, but in that case, too, gradual spindown will let the pulsar become invisible on a time scale that is short compared to the age of the galaxy.

There are two ways to make those extreme objects visible. There is an interesting parallel here with particle physics, where exotic particles are also studied using two ways, either by letting them collide with each other, or by studying their bound states. In astrophysics, collisions between stars can make otherwise invisible objects light up. In addition, a binary star containing one normal star and one extreme object can produce bright X rays when matter from the normal star falls onto the compact object.

Most stars in the Universe never interact very strongly with other stars, at least during their adult life, after they have left the interstellar gas cloud that was the nest in which they were born. However, there are various ‘dense stellar systems’, such as globular clusters and galactic nuclei, that contain stars that are sufficiently close to their neighbors to make collisions quite likely. By modeling the structure of such dense stellar systems, and comparing the modeling results with observations, we can gain valuable information about the nature of extreme objects, such as neutron stars and black holes. For example, only by studying the full ecology of a dense star system can we interpret the properties of the bound states between stars, in the form of binaries that may contain compact objects.

In addition to such stellar-mass objects, formed as byproducts of stellar evolution, many galaxies contain far more massive black holes at their centers. Our own galaxy contains a central black hole with a total mass close to 3.6×10^6 solar masses. This is a rather modest central black hole. Some galaxies contain holes that are more than a billion times more massive than our Sun. Such a central black hole becomes detectable only through interactions with the environment. Gas that is lost from nearby stars, or even stars plunging into such a supermassive black hole, can produce radiation in the X ray range as well as other wave length bands. In addition, a sufficiently massive black hole also affects the distribution and kinematics of the stars around it. Either way, the study of dense stellar systems is important for interpreting the observations of galactic nuclei.

2.3. Classical Physics Perspective

The gravitational N-body problem is the oldest unsolved problem in physics. After Newton formulated classical mechanics, and solved the two-body problem, attempts to solve the three-body problem did not lead to any practical form of a general solution. Progress in exploring the properties of the N-body problem had to wait till computers were available to do the very intensive number crunching required.

For the general gravitational N-body problem, we still cannot follow the complete evolution beyond values of N around $1 - 2 \times 10^5$. In order to solve the million-body problem, we will have to wait till the end of the next decade.⁴⁾ The maximum number of particles that we have been able to simulate in full glory started as $N = 10$

around 1960, and has been growing roughly according to Moore's law, when taking into consideration that the costs of a full-fledged N -body simulation has a scaling that is somewhat worse than $\propto N^3$. As a result, progress from $N = 10$ to $N = 10^6$ implies an increase in computational requirements of much more than a factor 10^{15} , corresponding to the change from kiloflops in 1960 to Exaflops, ten or more years from now.

These small N values may come as surprising news, given the many reports in the literature of N -body calculations with billions of particles in the case of cosmological simulations, and tens or hundreds of millions of particles in the case of galactic dynamics. The reason is that the latter two types of calculations are special, in that they use a softened approximation for the gravitational interactions, ignoring the singular nature of close encounters, and that cosmological simulations do not span many time steps (the Universe is dynamically young).

In the case of the general gravitational N -body problem, we start with an arbitrary configuration of N stars, which equilibrate in a few crossing times. After having reached dynamical equilibrium, further evolution takes place on a thermal time scale, through heat exchange through two-body relaxation effects, as in the molecular dynamics of the atoms in a gas. Since the effects of two-body encounters diminish with respect to the effects of the background potential of a star cluster as a whole, the N^2 computational load of following the interactions during one dynamical time scale is multiplied by another factor of N to form the scaling of roughly N^3 , alluded to above.

2.4. Computational Physics Perspective

The challenge of simulating a dense star cluster with a million stars is formidable, because of the enormous ranges in spatial and temporal scales that have to be modeled simultaneously. The size of a globular star cluster is measured in tens of parsecs, while the diameter of a neutron star is measured in kilometers, a discrepancy in distance scales of a factor 10^{15} . The time scale problems are even worse. The duration of a close passage of two neutron stars is measured in fractions of milliseconds, while the evolution of a star cluster can be comparable to the current age of the Universe, more than ten billions years, resulting in a discrepancy of time scales of a factor 10^{21} .

In order to make it possible to simulate a star cluster for ten billion years, it is necessary to introduce algorithms based on individual time steps, an approach pioneered and developed in great detail by Aarseth.⁵⁾ An analysis of the scaling of the computational cost of the general N -body problem was provided by Hut, Makino & McMillan,⁶⁾ who showed that for $N = 10^5$, direct N^2 methods are preferred. In order to reach $N = 10^6$, various algorithms can be employed to make a switch from N^2 to $N \log N$ scaling of inter-particle interactions, using tree methods such as introduced by Hut & Barnes.⁷⁾

Introductory material, as well as some new ideas about using a four-dimensional space-time perspective, can be found on the website of the *Art of Computational Science*.⁸⁾ There, a switch in perspective is presented from a notion of N bodies interacting in space to a collection of N world lines in spacetime, the configuration of which can be computed in a partially asynchronous way, using not only individual

time steps, but even individual algorithms.

In addition to the many algorithmic developments, significant speed has been gained by the construction of special-purpose hardware, in the form of the GRAPE family.⁹⁾ With a cost-performance ratio that is one or two orders of magnitude better than that of commercial supercomputers, the GRAPEs have dominated simulations of dense stellar systems for the last decade.

2.5. Interdisciplinary Physics Perspective

A detailed study of dense stellar systems requires the collaboration of astrophysicists with widely different specializations and backgrounds. Besides the multi-scale challenges summarized above, there is the multi-physics challenge of simultaneously modeling the physical evolution of individual stars, the hydrodynamical interactions between neighboring stars, and the gravitational interactions of the star cluster as a whole.

Whenever two or more stars approach each other closely, they can no longer be treated as point masses. Hydrodynamical calculations have to be employed to study the deformations and exchange of energy and angular momentum, and perhaps mass transfer or even a complete merger between the stars. Following those dynamical events, on a time scale of hours and days, the stars have to be followed for far longer time scales, of order of thousands if not millions of years, to follow the restoration of internal thermal equilibrium;

None of these treatments form part of the standard tool set of stellar dynamics, stellar evolution, or stellar hydrodynamics. New ideas need to be developed, together with new techniques and new implementations. It is in this area that there is plenty of room for basic breakthroughs, as mentioned in the first section of this paper. The MODEST initiative, for MOdeling DEense STellar systems^{10), 11)} has been organizing dozens of workshops to guide these developments, since its inception in 2002.

§3. Emission from Compact Objects

In the Universe there exist extreme objects that we cannot study in our laboratories: black holes and neutron stars. Although both have masses comparable to or moderately larger than the mass of the Sun, their ‘sizes’ are extremely small; only on the order of 10 km or so. A big distinction between them is that a neutron star has a solid surface, while a black hole has not. Neutron stars can support themselves by degeneracy pressure of neutrons against self-gravity, whereas black holes are collapsed objects because no counteracting force is strong enough to counter their gravity.

Why are we so much interested in such compact objects? There are several answers possible to this question, but a primary reason would be that we can get information as to the extreme physics, physics of extremely high density and high temperature material, sometimes with extremely strong magnetic fields, in extremely large gravitational fields, through the study of the compact objects. The existence of such extreme objects makes our view of the Universe remarkably rich. we can find these objects in dense stellar systems, and in turn we can also use these compact

objects as a probe to study the extreme conditions in dense stellar systems.

Then, how to detect compact objects? The most efficient way to identify neutron stars or black holes is to detect X-ray emission. This is because accretion onto a region with a dimension of $r_* \sim 10^6 \text{ cm}$ will emit strong X-rays. If the typical luminosity of $L \sim 10^{37-38} \text{ erg s}^{-1}$ is emitted as blackbody radiation from the area of $4\pi r_*^2$, the blackbody temperature will be $[L/(4\pi r_*^2 \sigma)]^{1/4} \sim 10^7 \text{ K}$, (where σ is the Stefan-Boltzmann constant) which implies the emission of X-rays. (If neutron stars have strong magnetic fields radio emission is also very important, in addition to X-ray emission.) If some source emits intense X-ray (and radio) emission and if the emission region is compact, it is likely to be an X-ray binary system, where a normal star loses gas that accretes onto the compact object. Let us see, next, some more details of emission properties from black holes and neutron stars, separately.

3.1. Black Holes

Since a single black hole cannot shine without environmental gas, we focus our discussion on the cases of binaries containing black holes, called black-hole binaries (BHBs). Spectral properties of BHBs have been investigated rather extensively recently and a variety of spectral states have been recognized. The most well-known spectral states are the so-called high-soft state and the low-hard state.¹²⁾¹³⁾

In the high-soft state, the disk spectra are blackbody and can well be represented by the standard disk model.¹⁴⁾ For a given mass M of a black hole and rate \dot{M} of mass accretion onto a compact star, the blackbody temperature (T_{disk}) of an accretion disk at a distance r from the black hole is given by the following relation:

$$\sigma T_{\text{disk}}^4 = \frac{3}{8\pi} \frac{GM\dot{M}}{r^3} \left(1 - \sqrt{\frac{r_*}{r}}\right), \quad (3.1)$$

where r_* is the radius of the inner edge of the disk. It is easy to show that the disk temperature reaches its maximum at $r = (7/6)^2 r_*$ and that the maximum temperature is about $\sim 10^7 \text{ K}$, as expected. The standard-type disks are optically thick, thus emitting blackbody radiation. Thus, the soft-state spectra are a sum of the blackbody radiation spectra with multiple temperatures, since the disk temperature is a function of radius. The flux will show an exponential roll-over at a frequency, corresponding to the maximum disk temperature, $\sim 10^7 \text{ K}$.

When the luminosity is less, say, $L/L_E < 0.03$, where $L_E [= 1.3 \times 10^{38} (M/M_\odot) \text{ erg s}^{-1}]$ is the Eddington luminosity, the spectra become significantly harder (with strong hard X-rays). This state is called a low-hard state. Typically, the disk spectra in the low-hard state are modeled by a cut-off power-law; $f_\nu \propto \nu^{-p} \exp(-h\nu/E_{\text{cut}})$, where p (~ 1.7 typically) is a constant called the spectral index and E_{cut} ($\sim 100 \text{ keV}$, which corresponds to a temperature of 10^9 K) is the cut-off energy. The most promising model explaining the physical situation of accretion disks in the low-hard state is the so-called RIAF (radiatively inefficient accretion flow) model.¹⁵⁾¹⁶⁾ According to the RIAF model, the disk (or flow) is not dense enough to emit substantial radiation. Then the disk gas does not cool efficiently, and gets hotter and hotter as one approaches the central black hole. As a result, the disk expands vertically and becomes low-density.

In addition to the two usual states mentioned above, two more distinct spectral states are known. In the very high state, both blackbody and power-law components are clearly present with rough equality. In the slim-disk state¹⁷⁾ which appears at even higher luminosities comparable to or even exceeding the Eddington luminosity^{*)}, again disk spectra are of blackbody type but with flatter temperature profiles

To summarize, we can roughly specify the black-hole mass (or L_E) through the X-ray spectral shapes of the BHBs. X-ray observations with good sensitivity are of great importance for this reason.

3.2. Neutron Stars

We can classify neutron stars into two categories according to their energy sources: rotation powered neutron stars and accretion powered neutron stars. If neutron stars have strong magnetic fields and rapid rotations they can emit periodic radio emission by magnetic dipole radiation. These are observed as pulsars. The typical magnetic field strengths of pulsars are $B \sim 10^{12}$ G. However, recently pulsars have been found those with extremely large magnetic field strengths, $B \sim 10^{15}$ G; these are called magnetars. They are bright in X-rays and occasionally emit bursts of $\sim 10^{41}$ erg s⁻¹ or more in X-ray and gamma-ray ranges.

The other category, accretion-powered neutron stars are found in X-ray binaries. Typically, there are two spectral components, both of blackbody type, observed in X-ray ranges: emission from an accretion disk and from the surface of a neutron star. (Note that a companion star is normally bright in the optical and infrared bands.) The former shows a spectrum of around 1 keV (corresponding to a temperature of $\sim 10^7$ K) and the latter of ~ 2 keV.

The virial theorem tells us that half of the gravitational energy that is released from gas reaching the neutron-star surface at r_* goes to kinematic (rotation) energy, while the remaining half goes to radiation from the disk surface. Therefore, the temperature differences of both components can be simply understood in terms of the different size of the emitting surface; that is, the neutron star surface is hotter than the hottest part of the disk because of its smaller surface area.

§4. Cosmic Laboratories

The study of dense stellar systems can be taken a long way with the classical gravitational N-body model. True, stars are not point-masses, but this idealization remains the basis of much of our understanding. This is the province of *stellar dynamics*.⁴⁾¹⁸⁾¹⁹⁾ In particular we are in a regime somewhere between $N = 2$ and $N = \infty$: the first case can be solved exactly, while in the second limit a stellar system behaves in most respects like a continuum. Outside astrophysics, the closest analogy to the regime of interest in this review is a collisional unmagnetized plasma, with the added complication of spatial inhomogeneity.

What makes the problem tractable is the empirical fact that a stellar system

*) The maximum possible luminosity for a spherically accreting system. No accretion is possible above this limit, since then radiation pressure force overcomes gravitational force.

of negative total energy is found to settle down reasonably quickly to a quasi-equilibrium structure. We shall consider the simplest, spherically-symmetric case. An application of the virial theorem then shows that, in an average sense,

$$2T + W = 0,$$

where T, W are, respectively, the kinetic and gravitational potential energies of the system. In this situation we can define a length scale for the system, R , called the “virial radius”, by the equation

$$W = -GM^2/(2R),$$

where M is the mass of the system, and G is the universal constant of gravitation. R is of the order of a parsec for star clusters. This relation holds for all isolated self-gravitating systems, stars as well as star clusters and galaxies.

In this “virial equilibrium”, the time scale on which stars orbit within the cluster is of order $2R/v$, where v is the root mean square speed of the stars (which is expressible in terms of M and T .) This time scale is called the “crossing time”, denoted t_{cr} , and is of the order of a million years for star clusters.

Virial equilibrium is a dynamical equilibrium, but not (in an appropriate sense) a *thermal* one. Just like atoms in a gas, stars in a cluster can exchange energy, though they do so by gravitational encounters only (at least, in the idealization of point masses). The time scale on which they do so is called the *relaxation time*, and it is generally much longer than the crossing time. In fact in virial equilibrium it is given to order of magnitude by

$$t_r \simeq 0.1Nt_{cr}/\ln N,$$

where N is the number of stars.

On this time scale, one would naively imagine that a stellar system can reach thermal equilibrium, the mean square velocities of the stars being the same everywhere, and inversely proportional to the stellar mass. Actually the thermodynamic study of this problem is a fascinating one,²⁰⁾ though it is complicated by the fact that stellar systems are “open”. It is true that gravitational encounters have the tendency to promote thermal equilibrium, but the fact that these systems are self-gravitating makes the results counter-intuitive.^{21) 22)} The fact that systems are open implies that stars escape, and so can never fully populate a Maxwellian velocity distribution. The fact that they are self-gravitating implies that, in the exchange of thermal energy, they behave in some respects as if they had negative specific heat. Therefore (though the argument is a little complicated²³⁾), stellar systems tend to form a hot, dense core, in a process referred to as “core collapse”. Its time scale is typically $15t_r$ if all stars have the same mass.

This is an unrealistic idealization, however, and we consider now the effects of the presence of particles with a variety of masses, which exhibit a effect of the tendency to equipartition of velocities: the heavier stars tend to lose kinetic energy in encounters, and sink into the core,²⁴⁾ and they do so on a time scale of order $t_r < m > /m_{max}$, where $< m >$ is the mean mass, and m_{max} is the mass of the heavier stars.

In real stellar systems, many stars are double (like a gas consisting of a mixture of single atoms and diatomic molecules). Three- and four-body interactions between these components introduce a new feature: a tendency (in interactions involving close pairs) for the kinetic energy of the products of an encounter to exceed that of the original participants. These interactions behave like an exothermic chemical reaction, heating up the gas of stars. This brings the process of core collapse to a halt (in systems with stars of a single mass), and has a similar effect on the segregation of heavy stars toward the center of the system. This puts the system at last onto a kind of thermal equilibrium, rather like that of a star, in which the central production of energy supports and powers the entire system, whose stars gradually evaporate, like enormous photons, into the environment of the stellar system.

To add realism to this fascinating but still idealized picture, we have to add the details of the way in which individual stars evolve internally while these dynamical processes are taking place. We do so in the context of astrophysical systems where this interplay is occurring before our eyes.

In the vicinity of the Milky Way galaxy are several places where the interaction between stellar and dynamical evolution has made its mark. Best studied of all are the old globular star clusters, because they are the closest and least obscured. Somewhat further away is the Galactic Center, though detailed observations became possible only recently because it is heavily obscured by dust and gas. Still further off are the young dense star clusters where stars have recently formed. Examples also occur in the Milky Way but again behind heavy layers of obscuration.

4.1. *The Galactic Center*

The center of our galaxy lies at a distance of about 8 kiloparsecs in the southern constellation of Sagittarius. It is marked by a bright radio and X-ray source, Sagittarius A*, which is now widely considered to be a black hole with a mass of approximately $3 \times 10^6 M_\odot$. It is surrounded by a “nuclear star cluster” in which the stellar density is well approximated by a broken power law with a break radius at about 0.22 parsec from the black hole.²⁵⁾ At this radius the stellar density is about $3 \times 10^6 M_\odot/\text{pc}^3$, and an additional comparable mass density is thought to be present in the form of stellar remnants (black holes, neutron stars, white dwarfs).

At the break radius, the time of relaxation (which is the time scale on which two-body gravitational encounters are effective) is already less than 1 Gyr,²⁶⁾ or about one tenth the age of the universe. Physical collisions become important on this time scale only much closer to the black hole, at a distance of order 0.02 pc.²⁷⁾ The short two-body relaxation time suggests that heavier stars near the galactic center should have sunk toward the center relative to lighter stars, by a process called “mass segregation”. (If the velocity dispersion of stars is independent of mass, an encounter between stars of different mass tends to leave the more massive star with a smaller velocity, which causes it to move onto a smaller orbit.) Indeed it is found that stars very close to the black hole are very massive, but they are also very *young*. Therefore they have not existed for long enough, compared with the relaxation time, to exhibit mass segregation. They must have appeared close to the black hole by some other processes.

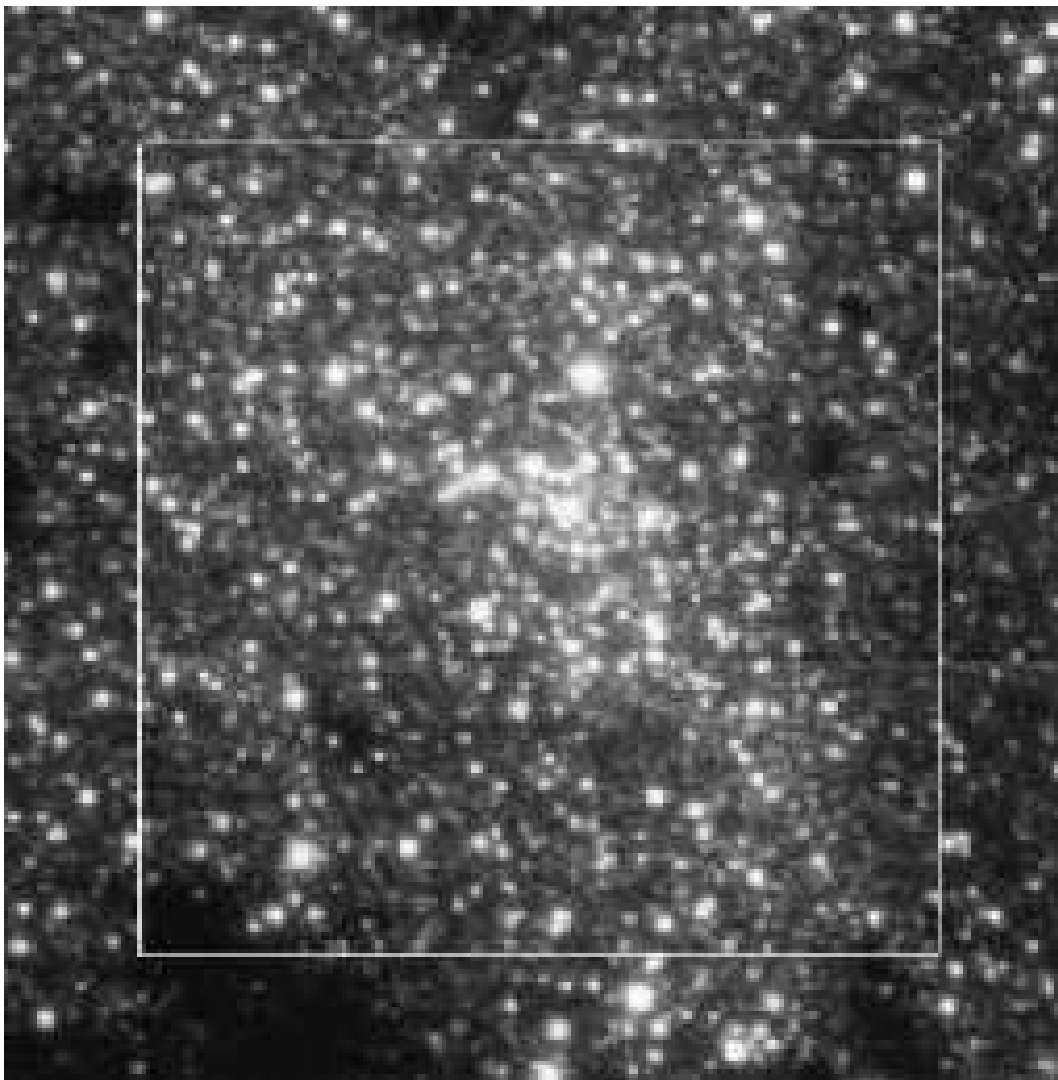


Fig. 1. The star cluster at the Galactic Center.²⁸⁾ The width of the outer frame is about 3 pc.

They could have formed *in situ*. But the enormous tidal gravitational field in the vicinity of the black hole should be an insurmountable obstacle to star formation. Perhaps they arose by collisions between more normal stars, which coalesce, resulting in a more massive, young-looking star? As we shall see, however, at the velocity dispersion found near the galactic center collisions destroy stars and do not make heavy stars. Another possibility is that the stars were born further away from the galactic center and rapidly migrated there. For example, a binary star may pass close to the black hole, which could disrupt the binary, and leave one star orbiting in the vicinity of the black hole. Indeed, at larger distances from the galactic center, where star formation presumably *is* possible, there are a few massive young star clusters. As these evolve and disrupt they tend to move toward the galactic center

(by a process akin to mass segregation), and could deliver young stars to its vicinity, though it is hard to see how this can be done quickly enough.

There is another possible explanation for the origin of S stars which simultaneously accounts for another puzzle about the stellar populations there: the observed paucity of red giants (relative to other types of stars) within about 0.2 pc of the galactic center. While collisions between binary stars and red giants may play a role,²⁹⁾ another possibility is that the envelopes of red giants are stripped off in encounters with the black hole. If so, something resembling an S star might result.³⁰⁾

There is another effect of all this dynamics near the black hole. Whether or not disrupted binaries can account for the S-stars, stars emerging from disruptive encounters can leave the galactic center with high speeds, much higher than the speeds of any other single stars in the Milky Way. Indeed in recent years several such high-speed stars have been discovered, though it is not clear that all of them can have emerged from the galactic center.

The Galactic Center can be studied in considerable detail because it is so close. Most galaxies are thought to harbour black holes, but their effects on stellar populations are much harder to detect. It has been proposed, however, that a recently discovered population of hot stars in the vicinity of the center of M31 (the Andromeda Galaxy) may share their origin with the S stars in the Galactic Center, controversial though that still is.

The Andromeda galaxy is more distant than the Galactic Center by a factor of about a hundred. Still further away are many examples of galaxies also thought with high probability to harbour black holes. It is impossible to detect the effects of these black holes on the stellar populations, except for the important dynamical effects on the space distribution and kinematic properties of the stars. But these black holes are an important indicator of the manner in which galaxies are formed, for it is found that there is a strong correlation between the inferred mass of the black hole and that of the galaxy itself (or, to be more precise, a particular component of it - the so-called “bulge”). The most massive galaxies contain the most massive black holes, with black hole masses up to around $10^9 M_\odot$.

4.2. *Globular Star Clusters*

The study of stellar populations in dense galactic nuclei is so hard because they are so distant or so highly obscured, or both. Globular star clusters are dense stellar systems which suffer from neither problem. They are smaller, mostly with less than a million stars each, but the closest one (the star cluster M4) is only about one quarter of the distance to the Galactic Center. These are the objects of choice for studying the effects of stellar density (number of stars per unit volume) on stellar populations. In the following, we discuss the exotic objects found in globular clusters, such as blue stragglers (4.2.1), X-ray binaries (4.2.2), unusual stellar populations (4.2.3) and black holes (4.2.4).

4.2.1. *Blue stragglers*

The first indications came over 50 years ago from studies of the colors and magnitudes of stars in individual clusters.³¹⁾ Each cluster appeared to consist of

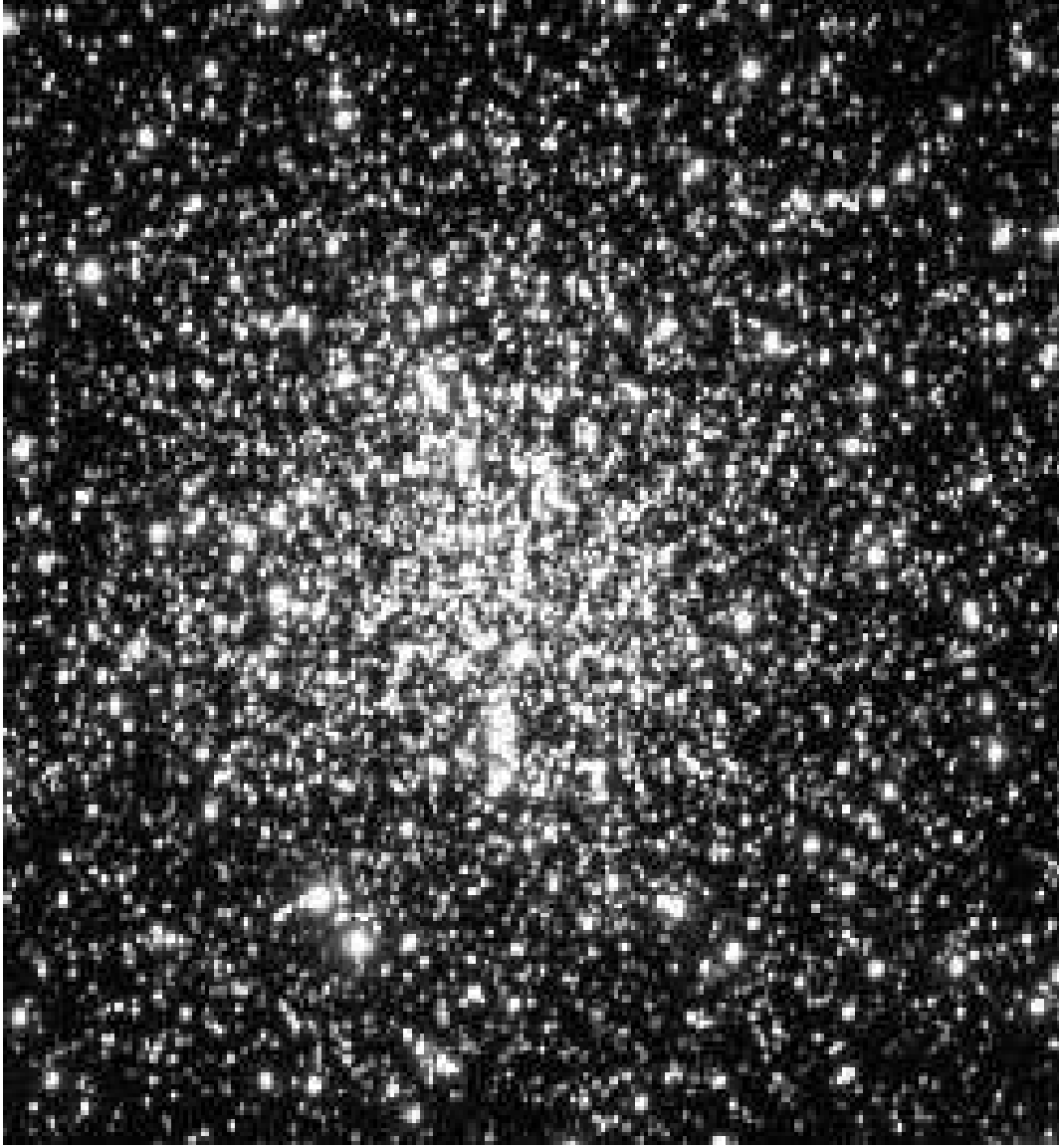


Fig. 2. The nearest globular star cluster, M4, which contains roughly 10^5 stars. The width of the frame is about 4 pc. (Credit NOAO/AURA/NSF)

stars of a unique age (comparable with the age of the Milky Way itself), except for a small number of “blue stragglers”. As their name suggests, they are hotter stars than the bulk of the stars in the cluster, and they *look* relatively youthful, to judge by their luminosity and color. In this respect they somewhat resemble the S stars near the Galactic Center, and they pose the same problems. Though they might have formed more recently than the other stars, there is no sign in any globular cluster of the gas clouds from which young stars can be born, and there are good reasons why such gas cannot exist in sufficient quantities. Unlike the S stars, however, there are

no essential difficulties in supposing that at least some of the blue stragglers are the result of collisions; the velocity dispersion in a globular cluster is much smaller than in a galactic nucleus, and colliding stars should coalesce with little loss of mass.³²⁾ The main difficulty is in establishing that collisions are common enough and the result will look like a blue straggler.

Estimates show that collisions in star clusters are rather common. In the globular cluster 47 Tucanae, for instance, perhaps as many as 1000 new systems (about 0.1% of all stars in the cluster) are produced by collisions between single stars in its lifetime.³³⁾ Not all of these are blue stragglers, but the numbers are enhanced by the behavior of *binary stars*. Once thought to be rare in globular clusters, it is now considered that their abundance is not much less than in the vicinity of the sun (i.e. over 50%). When one of the stars in a binary evolves, after consuming its central supply of hydrogen, it expands, and the two stars may well coalesce into a single star, which under suitable circumstances might be interpreted as a blue straggler.

Much effort is now being expended in conducting complete censuses of blue stragglers in globular clusters, to try to disentangle these two effects. Note that both are influenced by the dense stellar environment. For stellar collisions this is obvious, but also a dense environment may destroy the binaries which otherwise, in the course of time, would form blue stragglers. This explains why the number of blue stragglers in a cluster is found to be in anticorrelation with the number of collisions.³⁴⁾

One of the most intriguing problems posed by the collision hypothesis is how the collision remnant will evolve. At birth it is a highly unusual object, with an anomalous mixture of elements (depending on the state of evolution of the colliding stars), a high rotation rate (because of their orbital angular momentum), and high internal energy (from their original relative kinetic energy). This is one of the main motivations for trying to build a model of a star cluster which places dynamics, stellar evolution and collision hydrodynamics on an equal footing. All three ingredients have a strong role to play in determining the outcome of a collision.

4.2.2. X-ray sources

Historically the next indication that the stellar environment in a star cluster influences the stellar population there came from high-energy astrophysics. In the early days of X-ray astronomy it was found that sources were correlated with the positions of globular star clusters,³⁵⁾ a star in a cluster being about 100 times more likely to be an X-ray source than stars outside clusters. After two or three wrong turns, it was realized that these sources were binary stars containing a neutron star, the X-rays resulting from accretion of material from the outer layers of the binary companion. The reason why such objects are rare outside clusters is that a star in a binary is liable to destroy the binary when the star turns into a neutron star³⁶⁾ (at the end-point of its stellar evolution). In a cluster, however, it is relatively easy for a single neutron star to interact gravitationally with a binary (in a three-body interaction) in such a way that the neutron star displaces one of the original components of the binary.³⁷⁾ Such events would be extremely rare outside clusters. This remarkable overproduction of X-ray binaries in clusters has the effect that, when

one surveys nearby external galaxies, a large fraction of the bright X-ray sources one finds are associated with rich star clusters.³⁸⁾

There is another, weaker, class of X-ray sources (called *cataclysmic variables*) in which the role of the neutron star is taken by a white dwarf. Gravitational encounters are again invoked to explain their numbers, and quite recently this explanation was rather directly confirmed by the discovery of a strong correlation between the numbers of discrete X-ray sources in a cluster and the number of gravitational encounters.³⁹⁾

4.2.3. Anomalous populations

Another class of “star” found in extraordinary abundance in globular clusters are pulsars (neutron stars), especially those of the “millisecond” variety (i.e. the most rapidly rotating, as the name suggests).⁴⁰⁾ Again the process of capture of a neutron star by a normal binary is the key to understanding why they are found predominantly in places where the stellar density is high.⁴¹⁾

While various details of the populations of blue stragglers, X-ray sources and millisecond pulsars remain to be clarified, the broad outlines are clear. But there are other features of stellar populations in star clusters which are not understood at all. Some time ago it was found that there are color gradients in some clusters, the center being bluer than the periphery, and it was established that this was associated with a depletion of red giant stars near the center and with the density of the central “core” of the cluster.⁴²⁾ Perhaps this is due to the fact that encounters between stars are sufficiently frequent that the weakly bound envelopes of red giants are stripped,⁴³⁾⁴⁴⁾ but the argument has not been made quantitative and the problem itself has fallen into neglect.

Another feature which defies explanation is the behavior of the “horizontal branch”. This is a distinctive, bright and hot sub-population of evolved stars, which is found in all globular clusters. In some, however, this group of stars includes a number of faint and hot members called “faint blue horizontal branch stars”.⁴⁵⁾ Again they appear to correlate with the density of the central part of the cluster. Even more puzzling, however, is the recent discovery that the stars at the red extreme of this sequence are rotating very fast.⁴⁶⁾ The problem of explaining this peculiar population is wide open, but one possible explanation of fast rotation is coalescence following a collision (see the discussion of collisions in Sec.4.2.1).

4.2.4. Black holes

Finally we turn to the role of black holes. Stellar-mass black holes form in globular clusters as a natural end-product of the evolution of massive stars. They probably have a role to play in shaping the core of a star cluster (which may be defined as the region near the center, where the stellar density is within a factor two, in projection, of the central value).⁴⁸⁾⁴⁷⁾ But here we are thinking of *intermediate-mass* black holes, a putative class of black holes with masses between those of stellar remnants and those of the supermassive black holes in galactic nuclei; that is, black holes with a mass of order $1000M_{\odot}$.

One argument for supposing that such objects might exist in globular clusters is the known correlation between black hole mass and galaxy bulge mass in galaxies; if

this correlation is extended down to the mass of a globular star cluster, the inferred mass of a black hole would be of order $1000M_{\odot}$.⁴⁹⁾ It is not clear that globular clusters are a low-mass extension of galaxies, but a few of the most massive clusters (such as ω Centauri) are often interpreted as being the nuclear remnants of small galaxies.⁵⁰⁾⁵¹⁾

More direct evidence for intermediate-mass black holes in a few star clusters has come from measurements of the surface brightness and velocity dispersion of the stars, in much the same way that galactic supermassive black holes are identified. But the interpretation is complicated by the effects of mass segregation, which tend to modify these profiles in much the same way as a black hole does, and so the existence of an intermediate mass black hole is not uncontroversial in any cluster.⁵²⁾⁵³⁾⁵⁴⁾

A third form of evidence comes from rich star clusters of a very different kind. These are the young dense star clusters (dubbed yodecs) which are found in abundance in some kinds of galaxies (interacting galaxies, starburst galaxies) where the formation of stars is taking place at much higher rates than in the Milky Way (e.g.⁵⁵⁾). A few of these are found to be X-ray sources with a luminosity which suggests that the accreting object is a black hole (rather than a neutron star or white dwarf).⁵⁶⁾

The next question to be addressed is how such black holes can arise. Computer Modeling of yodecs has delineated the conditions under which this appears to be possible.⁵⁷⁾⁵⁸⁾ In its simplest terms, the condition is that a star must be able to collide and coalesce with other stars on a time scale no longer than the time of evolution of a massive star (i.e. a few million years). Then a very massive star may build up by a process of runaway coalescence. This condition on the collision rate requires a very high space density of the stars, but it appears that it can be satisfied in the densest yodecs, especially if the dynamical segregation of high mass stars is taken into account. The most uncertain step in this scenario for the formation of intermediate mass black holes, however, is what happens next: does an extremely massive star give rise to an intermediate-mass black hole?⁵⁹⁾⁶⁰⁾

This review of dense stellar systems has now come full circle, because one object which may harbour an intermediate mass black hole is group of stars close to the Galactic Center. These stars may be the most massive members of a cluster, and their velocity dispersion implies that, if they are indeed gravitationally bound, the presence of an intermediate mass black hole is indicated.⁶¹⁾⁶²⁾ Certainly, elsewhere in the vicinity of the Galactic Center there are a number of bona fide star clusters.⁶³⁾ During their short lives they spiral toward the Galactic Center itself. If some of these also contain intermediate-mass black holes, they may contribute to the build-up of the central black hole itself.⁶⁴⁾⁶⁵⁾

4.3. *Stellar Collisions*

The foregoing review of cluster dynamics focused on issues involving three processes: stellar evolution, stellar dynamics, and the non-gravitational interactions between stars. Now that the study of such interactions has been motivated, we shall describe in a little more depth just what is involved in modeling this problem. Of the three processes, it is the least well developed.

Long ago Spitzer and Saslaw⁶⁶⁾ proposed a model which works rather when the

relative velocity of the two interacting stars is high (relative to the escape speed from their surface). Roughly speaking, the parts of the stars which overlap (when viewed along the direction of their relative motion) is assumed to be lost if its total energy (calculated using momentum and energy conservation) is positive. The condition of high relative velocity is reasonably well satisfied in galactic nuclei.

For low-velocity encounters, such as generally occur in globular star clusters, the best developed analytical approach is one due to Lombardi and his colleagues.⁶⁷⁾ First the entropy profiles of the two stars are calculated, and then their union is sorted by specific entropy to construct the radial profile of the coalesced star.

Beyond these two approaches, hydrodynamical simulations are the method of choice, and again there are two approaches. One is through the use of grid-based hydrodynamic codes (as used in this problem, for example, by Ruffert⁶⁸⁾), and the other, now perhaps more comprehensive, is Smoothed Particle Hydrodynamics. The latter approach has been developed over many years in the astrophysics community and several sophisticated aspects of its behavior are now well established in various contexts.^{69) 70)}

The result of an encounter depends in an essential way on several factors: the masses of the two stars, their stellar type (red giants and neutron stars behave in very different ways, for example), their relative speed, and their distance of closest approach. Extensive libraries of collision data can now be accessed,⁷¹⁾ but there are other important parameters which render the tabulation of collision outcomes increasingly difficult, e.g. stellar rotation. For this reason, there is a real need to incorporate live SPH collision simulations into a comprehensive program for simulating all essential aspects of dense stellar systems (Sec.5).

Collisions between stars are rare events, even in dense stellar systems. Recently, however, an unusual observation was made that could possibly be caused by a stellar collision, in the galaxy M85, in the Virgo cluster.⁷²⁾

§5. Virtual Laboratories

Astrophysics is the only field within physics that has no laboratory component. Within the immediate neighborhood of the Earth, in our own solar system, we have access to meteorites that reach the Earth and samples that have been returned from the Moon or that have been analyzed *in situ* by robotic explorers on planets or moons in our solar system. But anything outside our own planetary system is completely outside our reach.

Until half a century ago, astronomy thus had the strange distinction of being at the same time the oldest modern science, giving rise to classical mechanics, but also the only modern science without a lab. Happily, this changed with the advent of electronic computers, which have provided astrophysicists with a virtual laboratory in which to conduct experiments.

5.1. Early History

The first experiments conducted in these laboratories focused on the evolution of single stars. Models of stars such as our own Sun were constructed, and they were

evolved during the billions of years of their total life time. Within a decade, the field of computational stellar evolution matured enough to see the publication of several text books.

With a somewhat later start, in the early sixties, simulations of the gravitational interactions between stars in star clusters took off. This field of stellar dynamics is similar in many respect to that of molecular dynamics, in the classical approximation. In both cases, interesting results can be obtained in the point-particle approximation, and the main difference is the use of an attractive inverse square force law to model gravity, versus more complex laws such as those based on a Lennard-Jones potential in the case of molecules.

It is an interesting question why these two fields, stellar evolution and stellar dynamics, remained relatively separate for several decades. Only in the last ten years have stellar dynamics simulations included detailed recipes for the evolution of single stars and double stars, and a real coupling of ‘live’ stellar evolution codes and stellar dynamics codes has only begun last year, with the work of Church.¹⁾

A third area that is relevant is that of stellar hydrodynamics, describing what happens when two stars physically collide. The very first calculations in this field were also done in the sixties, but again, stellar hydrodynamics did not become integrated with stellar evolution and stellar dynamics for a long time. In fact, we are still waiting for the first such combined simulations.

5.2. *Multi-Scale Challenges*

In the introduction, we have already mentioned the vastly different spacetime scales on which we have to model the evolution of a star clusters. Spatial scales range over 15 orders of magnitude, from kilometers to parsecs, and temporal scales range over 21 orders of magnitude from fractions of milliseconds to many billions of years. As a result, it is impossible to follow the evolution of a star cluster using standard text book numerical integration schemes, even if we were to approximate stars as point particles.

To see this, let us make a simple estimate of the computational needs for a straightforward stellar dynamics simulation. In a globular cluster with a million stars, there are 10^{12} pair-wise gravitational forces that we have to consider. If we were to evaluate each pair-wise force, on the shortest time scale that configurations can change, we would need to repeat that exercise 10^{21} times. This implies that we need to calculate 10^{33} pair-wise force calculations. A typical calculation involves a few dozen floating point calculations, so the total cost would exceed 10^{34} floating point calculations. Even with a future supercomputer speed of 1 Petaflops, or 10^{15} floating point calculations per second, a simulation would take 10^{19} seconds, or more than 10^{12} years, a hundred times longer than the current age of the universe.

Of course, it would be an enormous waste of time to model a star cluster in such a way, using constant time steps. A much better approach would be to use adaptive time steps. At each moment, we can determine which stars are involved in a relatively close encounter, and we can then enforce a system timestep of the appropriate size. Sometimes such a timestep will be a fraction of a millisecond, but at other times the closest stellar passage may take place on time scales of seconds or minutes, thereby

speeding up the calculations by several orders of magnitudes. However, this in itself will not beat the factor of 10^{12} that separates us from a brute-force calculation and the requirement to finish a simulation in at most a year.

Unfortunately, most standard numerical methods textbooks do not go further than recommending adaptive time steps. In order to make stellar dynamics calculations of star cluster feasible at all, astrophysicists had to invent completely new integration schemes, together with all kinds of other specialized algorithmic tricks.

A very important step is the switch from shared adaptive time steps to individual time steps. When some stars undergo a particularly close encounter, there really is no reason to slow all other stars down to the same short time steps as is needed to resolve that encounter. Stars that are relatively more isolated can be allowed to take much longer integration time steps. The trick here is to use an extrapolation method to predict the position of those slow stars, to allow us to calculate their forces on those stars that move faster and in more irregular ways.

Another equally important step is to treat the evolution of tight clumps of small numbers of stars separately. A tight double star, for example, can be integrated analytically when the stars are so close together that the perturbations of all other stars can be safely neglected. And even when two stars are not that close, the perturbation of other stars can be taken into account in approximate ways, saving orders of magnitude in computer time. Similarly, triple stars that are almost isolated can be effectively frozen or put in a form of quarantine until neighboring stars come sufficiently close to play an important role.

For an overview of various classes of algorithms, see Aarseth.⁵⁾ For a detailed analysis of the computational costs of the main algorithms, see Makino & Hut.⁷³⁾⁷⁴⁾

5.3. *Multi-Physics Challenges*

5.3.1. The Basic Picture

It is easy to sketch a picture of how you could model the structure and evolution of a star cluster, while taking into account the structure and evolution of the individual stars that constitute the cluster. Since most stars are either single, or part of a binary in which the stars are not in very close proximity, those stars can be modeled as point particles, to a high degree of accuracy. For each such star, we can run a stellar evolution code to keep track of their internal properties, as well as their radius, but with respect to the dynamical interactions with other stars, their non-zero size does not need to be taken into account.

When two or more stars come close to each other, however, the point mass approximation breaks down. In such a case we can take the mass points, representing the stars involved in such an encounter, and replace them by hydrodynamical models. For example, we can use the Smooth Particle Hydrodynamics (SPH) approximation, which is natural since it is also particle based, but we could use other approaches as well.¹¹⁾ We can then model the subset of hydrodynamically realistic stars during the short time of their encounter, under the perturbing influence of nearby stars that are still treated as mass points.

On a time scale short with respect to the evolution of the whole star cluster, typically days or weeks, the two or more stars may merge, or separate again, or

perhaps settle into a contact binary configuration. In any of these cases, we can wait for dynamic equilibrium to be restored, and then we can replace the stars again by point masses, as far as the subsequent stellar dynamical evolution is concerned, with a special treatment for the higher-order multipoles of the binaries, where needed.

For the stars to regain their thermal equilibrium will require vastly longer than the days or weeks needed for dynamical equilibrium to be restored. Here we are talking about millions or perhaps hundreds of millions of years, depending on the mass of the stars. During this time we will have to run an active stellar evolution code to model the evolution of such an unusual star (for other, normal, stars we have the luxury of using table look-up methods, rather than evolving the star using a live stellar evolution code¹¹⁾).

5.3.2. Steps Toward Implementations

The picture sketched above may look simple in principle, but writing a computer code that can automatically take care of all the conversions and interactions is a challenging task. The logistics for treating encounters between single stars and/or binaries is already quite complicated. What is worse, there is a small chance that additional stars or binaries will show up, after the initial interactions have started. For example, when two binaries encounter each other, a third binary may approach the system, leading to a strongly interacting six-body system. To treat all possible types of outcome correctly is not an easy programming task.

An important step in the direction of constructing such a code was taken with the release of the Kira code, at the core of the Starlab environment.⁷⁵⁾⁷⁶⁾ In this case, stellar evolution was modeled through the use of recipes and fits to stellar evolution tracks. In the near future, actual stellar evolution codes will be connected to stellar dynamics and hydrodynamics codes. A step in that direction is currently being taken by the international MUSE collaboration,⁷⁷⁾ aimed at connecting existing codes in all three fields, using Python as a ‘glue’ language. The MUSE project is one of the main activities under the MODEST umbrella, short for MOdeling DEnse STellar systems.¹⁰⁾ A complimentary approach is made in the ACS project, short for the Art of Computational Science.⁷⁸⁾ There the emphasis is on writing extremely modular codes that have many levels of hooks for connecting with other modules that model different types of physics.

5.4. *GRAPE*

As mentioned in section 2.4, GRAPEs have been extensively used for the simulations of dense stellar systems. The first GRAPE hardware used for such simulations was GRAPE-2,⁷⁹⁾ with a peak speed of 40 Mflops. In 1995, the GRAPE-4 system, with a peak speed of 1.08 Tflops, was completed. It used 36 processor boards, each with 48 special processor chips. One processor chip delivered a speed of 640 Mflops. Thus, a single processor board had a speed of about 30 Gflops. In 2002 the GRAPE-6, with a peak speed of 64 Tflops, was completed. Its processor chip has a speed of 31 Gflops, and the peak speed of a single board with 32 processor chips is 1 Tflops. A 4-chip version of the GRAPE-6 was also developed. Many copies of GRAPE-6 boards were made, and they are used by many researchers, around the world.

Currently, the GRAPE-DR system is under development.⁸⁰⁾⁸¹⁾ Unlike previous versions of GRAPE hardware, which have processors specialized for the calculation of pairwise gravitational interactions between particles, GRAPE-DR integrates a number of very simple but programmable processors, and can be used for a much wider range of applications. The name GRAPE is for Gravity Pipe, but GRAPE-DR means Greatly Reduced Array of Processor Elements with Data Reduction. An obvious application for the GRAPE-DR processor are SPH calculations. A single GRAPE-DR processor chip delivers 512 Gflops, and a small 4-chip card will deliver 2 Tflops, or twice the speed of the large, 32-chip GRAPE-6 board.

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